IMM 347
February 1966

Courant Institute of
Mathematical Sciences

# Cnoidal Waves In a Toroidal Channel

A. S. Peters

Prepared under Contract Nonr-285(55)
with the Office of Naval Research NR 062-160

Distribution of this document is unlimited.

New York University

IMM-
347
c.2

NEW YORK UNIVERSITY
COURANT INSTITUTE - LIBRARY
251 Mercer St. New York, N.Y. 10012

NR 062-160

IMM 347
February 1966

New York University
Courant Institute of Mathematical Sciences

# CNOIDAL WAVES IN A TOROIDAL CHANNEL

A. S. Peters

This report represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the Office of Naval Research, Contract Nonr-285(55). Reproduction in whole or in part is permitted for any purpose of the United States Government.

Distribution of this document is unlimited.

c. 2

## 1. Introduction

In a recent paper the author [1] used a variant of Friedrichs' [2] technique of stretching in order to study irrotational and rotational solitary waves in a long straight channel with arbitrary cross section. The object of the present paper is to extend the ideas of [1] so that we can consider irrotational cnoidal waves in a gravitating liquid confined to a toroidal channel: a channel whose wall is formed by the rotation of an arc around a fixed axis as shown in Fig. 1 in Section 2. We suppose that the liquid is subject to a body force directed toward this axis. One of the reasons for considering an analysis of cnoidal waves in such a fluid body is that it may give some insight with respect to nonlinear waves in a liquid or gaseous ocean confined to a channel along a parallel of latitude on a sphere. It is not *a priori* clear that cnoidal or solitary waves exist in a toroidal ocean since their existence in straight channels is strongly coupled with the fact that waves of very large wave length exist in such channels. The infinite length of the straight channel, in which it has been proved that cnoidal waves can exist, stands in contrast with the finite length of the liquid filled toroidal channel which is the subject of the problem treated here.

Section 2 contains a formulation of our problem and we present there the equations which are produced when a stretching technique is applied to the hydrodynamical equations for the motion of an incompressible liquid in a toroidal channel when the liquid is subject to an axially symmetric gravitational force directed toward the axis of rotation of the channel.

In Section 3 we examine some phases of the linear theory of motion in the channel which theory is based on the assumption that the wave amplitude is small. This section contains a discussion of the speeds associated with the characteristic sinusoidal waves which are admitted by the linear theory.

Section 4 is an amplification of an analysis completed some time ago in order to answer certain questions raised by J. J. Stoker about two-dimensional nonlinear waves in a body of water whose equilibrium shape coincides with a hollow cylinder. We demonstrate in this section that a formal expansion can be chosen such that the first term, the lowest order approximation, represents a cnoidal wave profile. The first order speed associated with this wave is greater than the speed of any one of the characteristic sinusoidal waves of the linear theory. The representation for the free surface displacement $\eta$, and other representations which we give for the velocity components and pressure stem, as it turns out, from the classical equation

$$(1.1) \qquad m_0\eta_1'''(\sigma) = m_1\eta_1(\sigma)\eta_1'(\sigma) + m_2\eta_1'(\sigma) \ .$$

This is the well known equation which arises in all of the cases noted above. Under certain conditions it has a solution which can be expressed in terms of the cn function, a Jacobi elliptic function. We find that the coefficients of (1.1) depend on the solution $\psi_2(\rho,\zeta)$ of a boundary value problem of elliptic type. If the cross section of the channel is not a rectangle $\psi_2(\rho,\zeta)$ depends on both $\rho$ and $\zeta$ and hence the motion has three-dimensional aspects. We find in every case that the first order approximation $\eta_1$ to the free surface admits no change in height in a lateral direction across the channel. However, if the cross section of the channel is not rectangular, higher order approximations show that the height is not constant in the lateral axial direction. Thus the theory presented here leads to a three-dimensional wave.

These results and others are summarized and discussed in more detail in Section 5.

## 2. Formulation

Let $-\underline{r}(P)$ denote the vector directed from a point P to the nearest point Q which lies on a fixed $x_3$-axis. If $\delta$ is the density of a particle at P, and if r is the distance from Q to P, the vector $-\frac{g\delta}{r}\cdot\underline{r}(P)$ defines a force field which can be regarded as a gravitational field. We suppose that this field compels an incompressible, inviscid liquid, with constant density $\delta$ to remain in a toroidal channel. The inner wall of the channel is the surface generated by rotating the piecewise smooth curve

$$r = a + Y(x_3) = \alpha h + Y(x_3) \tag{2.1}$$

around the $x_3$-axis. The letter $\alpha$ denotes a free constant while h denotes a unit of length. A cross section of the channel lying in the $x_2x_3$-plane is shown in Fig. 1. The cross at the origin in the diagram indicates that the $x_1$-axis is directed into the paginal surface. We assume that the $x_1$, $x_2$, $x_3$-axes define an inertial rectangular reference frame. We suppose for simplicity that a line parallel to the $x_3$-axis intersects the channel wall in no more than two points; that $x_2 = \alpha h$ intersects the wall in the points $(h\zeta_1, \alpha h)$ and $(h\zeta_2, \alpha h)$ such that $b = \beta h = h(\zeta_2 - \zeta_1) > 0$, where b is the breadth of the equilibrium free surface. We also suppose that the maximum depth of the liquid in the channel is small compared to the distance of the bottom from the $x_3$-axis which we assume is large.

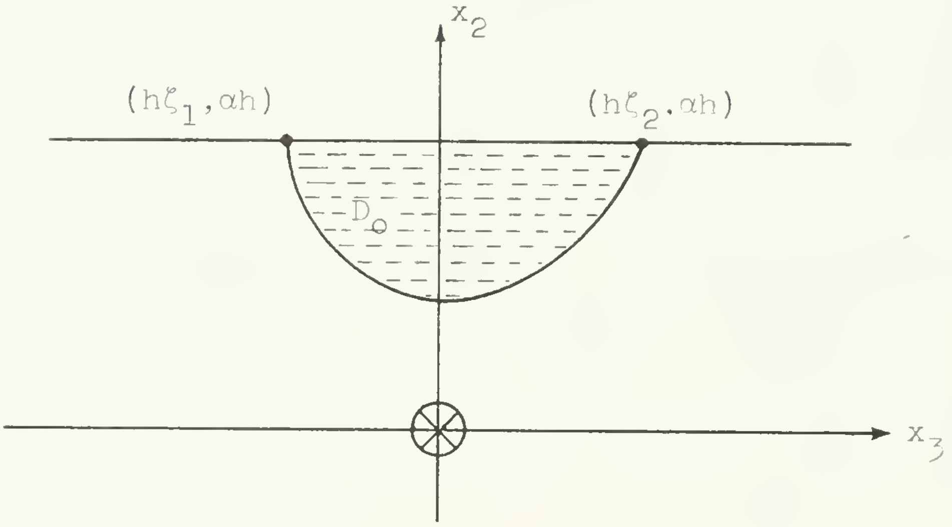

Fig. 1

We will use the half plane M defined by

$$x_2 = 0 \; ; \quad x_1 > 0$$

as the basis of a cylindrical coordinate system in which a point has coordinates $(r, \theta, x_3)$ where $\theta$ is the angle which the vector r̲ makes with M. The angle $\theta$ is to be measured from M in a counterclockwise direction — the direction defined by the palm directed fingers of the right hand when the thumb points in the direction of the positive $x_3$-axis. In terms of these coordinates the equation of the free surface of the liquid in the channel can be written as

$$r = a + f(\theta, x_3, t) \tag{2.2}$$

where t stands for time.

Let the cylindrical velocity components of a liquid particle be $v_1 = -r\,\frac{d\theta}{dt}$, the clockwise tangential speed; $v_2 = \frac{dr}{dt}$, the radial speed; and $v_3 = \frac{dx_3}{dt}$ the axial speed. If the gravitational force and the pressure p are the only forces acting on the liquid the elementary theory of hydrodynamics predicts that the motion of the liquid is determined by the continuity equation

$$-\frac{1}{r}\frac{\partial v_1}{\partial \theta} + \frac{v_2}{r} + \frac{\partial v_2}{\partial r} + \frac{\partial v_3}{\partial x_3} = 0 \; ; \tag{2.3}$$

the momentum equations

$$(2.4)\quad \begin{cases} \dfrac{\partial v_1}{\partial t} - \dfrac{v_1}{r}\dfrac{\partial v_1}{\partial \theta} + v_2\dfrac{\partial v_1}{\partial r} + v_3\dfrac{\partial v_1}{\partial x_3} + \dfrac{v_1 v_2}{r} = -\dfrac{1}{\delta r}\dfrac{\partial p}{\partial \theta} \\[2ex] \dfrac{\partial v_2}{\partial t} - \dfrac{v_1}{r}\dfrac{\partial v_2}{\partial \theta} + v_2\dfrac{\partial v_2}{\partial r} + v_3\dfrac{\partial v_2}{\partial x_3} - \dfrac{v_1^2}{r} = -g - \dfrac{1}{\delta}\dfrac{\partial p}{\partial r} \\[2ex] \dfrac{\partial v_3}{\partial t} - \dfrac{v_1}{r}\dfrac{\partial v_3}{\partial \theta} + v_2\dfrac{\partial v_3}{\partial r} + v_3\dfrac{\partial v_3}{\partial x_3} = -\dfrac{1}{\delta}\dfrac{\partial p}{\partial x_3} \end{cases}$$

the kinematic boundary conditions

$$(2.5)\qquad v_2 = v_3\,\frac{\partial Y(x_3)}{\partial x_3}\ ;$$

$$(2.6)\qquad v_2 = -\frac{v_1}{r}\frac{\partial f}{\partial \theta} + v_3\frac{\partial f}{\partial x_3} + \frac{\partial f}{\partial t}\ ;$$

plus dynamical conditions at the free surface; and initial conditions at $t = 0$. The vorticity vector is

$$\begin{aligned}(2.7)\qquad \underline{V} &= V_1\underline{k}_1 + V_2\underline{k}_2 + V_3\underline{k}_3 \\ &= \left(\frac{\partial v_3}{\partial r} - \frac{\partial v_2}{\partial x_3}\right)\underline{k}_1 + \left(\frac{1}{r}\frac{\partial v_3}{\partial \theta} + \frac{\partial v_1}{\partial x_3}\right)\underline{k}_2 \\ &\quad + \left(-\frac{v_1}{r} - \frac{1}{r}\frac{\partial v_2}{\partial \theta} - \frac{\partial v_1}{\partial r}\right)\underline{k}_3 .\end{aligned}$$

Here, $\underline{k}_1$, $\underline{k}_2$ and $\underline{k}_3$ are mutually orthogonal unit vectors which define a local coordinate system at any point P. The vector $\underline{k}_3$ has the same direction as the $x_3$-axis; $\underline{k}_2 = \frac{r}{r}$ is a radial vector, and $\underline{k}_1$ points clockwise.

The above equations can be written in dimensionless form if we introduce a typical length in the radial direction say h; a dimensionless stretching factor $\sqrt{\varepsilon}$; and the dimensionless variables

$$\sigma = \sqrt{\varepsilon}\ \theta\ , \quad \rho = \frac{r}{h}\ , \quad \zeta = \frac{x_3}{h}\ , \quad \tau = \sqrt{\varepsilon}\ (\frac{g}{h})^{1/2}\ t$$

$$u = \frac{v_1}{\sqrt{gh}}\ , \quad v = \frac{\sqrt{\varepsilon}\ v_2}{\sqrt{gh}}\ , \quad w = \frac{\sqrt{\varepsilon}\ v_3}{\sqrt{gh}}\ , \quad \pi = \frac{p}{\tau gh}$$

$$\alpha = \frac{a}{h}\ , \quad \eta = \frac{f}{h}\ , \quad y = \frac{Y}{h}$$

$$\nu_1 = (\frac{h}{g})^{1/2}\ \sqrt{\varepsilon}\ V_1\ , \quad \nu_2 = (\frac{h}{g})^{1/2}\ V_2\ , \quad \nu_3 = (\frac{h}{g})^{1/2}\ V_3$$

In terms of these variables the basic equations are

$$-\frac{\varepsilon}{\rho}\frac{\partial u}{\partial \sigma} + \frac{v}{\rho} + \frac{\partial v}{\partial \rho} + \frac{\partial w}{\partial \zeta} = 0\ ; \tag{2.8}$$

$$\left\{\begin{aligned} &\varepsilon[\frac{\partial u}{\partial \tau} - \frac{u}{\rho}\frac{\partial u}{\partial \sigma} - \frac{1}{\rho}\frac{\partial \pi}{\partial \sigma}] + v\frac{\partial u}{\partial \rho} + w\frac{\partial u}{\partial \zeta} + \frac{uv}{\rho} = 0 \\ &\varepsilon[\frac{\partial v}{\partial \tau} - \frac{u}{\rho}\frac{\partial v}{\partial \sigma} - \frac{u^2}{\rho} + 1 + \frac{\partial \pi}{\partial \rho}] + v\frac{\partial v}{\partial \rho} + w\frac{\partial v}{\partial \zeta} = 0 \\ &\varepsilon[\frac{\partial w}{\partial \tau} - \frac{u}{\rho}\frac{\partial w}{\partial \sigma} + \frac{\partial \pi}{\partial \zeta}] + v\frac{\partial w}{\partial \rho} + w\frac{\partial w}{\partial \zeta} = 0 \end{aligned}\right. \tag{2.9}$$

$$v = w\frac{\partial y}{\partial \zeta}\ ; \tag{2.10}$$

$$\varepsilon[\frac{\partial \eta}{\partial \tau} - \frac{u}{\rho}\frac{\partial \eta}{\partial \sigma}] = v - w\frac{\partial \eta}{\partial \zeta}\ ; \tag{2.11}$$

$$(2.12)\qquad \begin{cases} \nu_1 = \dfrac{\partial w}{\partial \rho} - \dfrac{\partial v}{\partial \zeta} \\[2ex] \nu_2 = \dfrac{1}{\rho}\dfrac{\partial w}{\partial \rho} + \dfrac{\partial u}{\partial \zeta} \\[2ex] \nu_3 = -\dfrac{u}{\rho} - \dfrac{1}{\rho}\dfrac{\partial v}{\partial \sigma} - \dfrac{\partial u}{\partial \rho} \end{cases}$$

When the motion of the liquid in the channel is irrotational there exists a velocity potential $\phi(\rho,\sigma,\zeta)$ such that for a liquid particle with coordinates $(\rho,\sigma,\zeta)$ the velocity components are

$$(2.13)\qquad u = -\frac{\phi_\sigma}{\rho}\ , \quad v = \phi_\rho\ , \quad w = \phi_\zeta\ ;$$

and, by virtue of the continuity equation, $\phi$ must satisfy

$$(2.14)\qquad \frac{\varepsilon\phi_{\sigma\sigma}}{\rho^2} + \frac{\phi_\rho}{\rho} + \phi_{\rho\rho} + \phi_{\zeta\zeta} = 0\ .$$

The momentum equations in terms of $\phi$ are

$$\begin{aligned} &\varepsilon[\phi_{\sigma\tau} + \frac{1}{\rho^2}\phi_\sigma\phi_{\sigma\sigma} + \pi_\sigma] + \phi_\rho\phi_{\rho\sigma} + \phi_\zeta\phi_{\zeta\sigma} &&= 0 \\ &\varepsilon[\phi_{\sigma\tau} + \frac{1}{\rho^2}\phi_\sigma\phi_{\sigma\rho} - \frac{1}{\rho^3}\phi_\sigma^2 + 1 + \pi_\rho] + \phi_\rho\phi_{\rho\rho} + \phi_\zeta\phi_{\zeta\rho} &&= 0 \\ &\varepsilon[\phi_{\zeta\tau} + \frac{1}{\rho^2}\phi_\sigma\phi_{\sigma\zeta} + \pi_\zeta] + \phi_\rho\phi_{\rho\zeta} + \phi_\zeta\phi_{\zeta\zeta} &&= 0 \end{aligned}$$

and these yield the energy integral

$$(2.15)\qquad \varepsilon[\phi_\tau + \frac{\phi_\sigma^2}{2\rho^2} + \rho + \pi] + \frac{\phi_\rho^2}{2} + \frac{\phi_\zeta^2}{2} = \text{const.}$$

This integral requires the dynamical surface condition

$$(2.16)\qquad \varepsilon[\phi_\tau + \frac{\phi_\sigma^2}{2\rho^2} + \alpha + \eta + \pi] + \frac{\phi_\rho^2}{2} + \frac{\phi_\zeta^2}{2} = \text{const.}$$

to be satisfied at the free surface $\rho = \alpha + \eta(\sigma, \zeta, \tau)$. In addition to this, equations (2.11) and (2.13) show that the kinematic condition

$$(2.17)\qquad \varepsilon[\eta_\tau + \frac{\phi_\sigma \eta_\sigma}{\rho^2}] = \phi_\rho - \phi_\zeta \eta_\zeta$$

must also be satisfied at $\rho = \alpha + \eta$. In terms of $\phi$, equation (2.10) is

$$(2.18)\qquad \phi_\rho = \phi_\zeta y_\zeta$$

which asserts that the normal velocity of the liquid is zero at the rigid channel wall $\rho = \alpha + y(\zeta)$.

The equations (2.8)-(2.18) are analogous to the equations which were recently used by the author [1] in order to study solitary waves in a horizontal straight channel with arbitrary cross section. The presumption here is that (2.8)-(2.18) will be useful for the investigation of nonlinear waves in a toroidal channel. However, before we examine this matter let us consider in the next section some linear aspects of the above equations.

## 3. Linear Theory. Irrotational Motion

For the purposes of this section no stretching is needed, that is, we take $\varepsilon = 1$ so that $\sigma = \theta$.

If the motion of the liquid in the toroidal channel is such that $u = u_o$, $v = v_o$, $w = w_o$ where

$$\frac{\partial u_o}{\partial \tau} = 0 \ , \quad \frac{\partial u_o}{\partial \theta} = 0 \ , \quad v_o = 0 \ , \quad w_o = 0$$

then the wall condition (2.18) is satisfied and we will say that the channel contains a liquid stream. As we can see from equations (2.12) a liquid stream is irrotational only if

$$u_o = \frac{\text{const.}}{\rho} = \frac{c}{\rho\sqrt{gh}} = -\frac{\phi_{o\theta}}{\rho} \ .$$

For this kind of motion equation (2.15) shows that the pressure is a function of $\rho$ only and (2.16) shows that the equation of the free surface is $\rho = \alpha + \text{const.}$ Since the constant $\alpha$ is arbitrary the free surface can be defined by $\rho = \alpha$ $(\eta = 0)$; and if we take the free surface pressure to be zero we have

$$\pi_o = \alpha - \rho + \frac{c^2}{2gh}\left(\frac{1}{\alpha^2} - \frac{1}{\rho^2}\right) \ .$$

Let us consider the problem of finding the irrotational steady motion which results when we apply a slight pressure $F(\theta,\zeta)$ to the surface of a liquid stream as defined above. The potential function for the disturbed stream can be expressed as

$$\phi = \phi_o + \phi^* = - \frac{c\theta}{\sqrt{gh}} + \phi^* ,$$

while the pressure and the free surface are respectively represented by

$$\pi = \pi_o + \pi^* = \alpha - \rho + \frac{c^2}{2gh} \left(\frac{1}{\alpha^2} - \frac{1}{\rho^2}\right) + \pi^*$$

and

$$\rho = \alpha + \eta^* .$$

Without any stretching, that is with $\varepsilon = 1$ in (2.14)-(2.18), the starred perturbation quantities must satisfy the following nonlinear system. The equations

$$\frac{\phi^*_{\theta\theta}}{\rho^2} + \frac{\phi^*_\rho}{\rho} + \phi^*_{\rho\rho} + \phi^*_{\zeta\zeta} = 0 \tag{3.1}$$

and

$$\frac{(\phi^*_\theta)^2}{2\rho^2} - \frac{c\phi^*_\theta}{\rho^2\sqrt{gh}} + \pi^* + \frac{(\phi^*_\rho)^2}{2} + \frac{(\phi^*_\zeta)^2}{2} = \text{const.} \tag{3.2}$$

must hold at any point occupied by a liquid particle while

$$\frac{1}{\rho^2} \left(\phi^*_\theta - \frac{c}{\sqrt{gh}}\right)\eta^*_\theta = \phi^*_\rho - \phi^*_\zeta \eta_\zeta \tag{3.3}$$

and

$$F(\theta,\zeta) = \alpha - \rho + \frac{c^2}{2gh} \left(\frac{1}{\alpha^2} - \frac{1}{\rho^2}\right) + \pi^* \tag{3.4}$$

must be satisfied at the free surface $\rho = \alpha + \eta^*$, and

$$\phi^*_\rho = \phi^*_\zeta y_\zeta \tag{3.5}$$

must be satisfied for $\rho = \alpha + y(\zeta)$. These equations can be linearized by neglecting products of starred quantities. If we do this we find that $\phi^*$ must satisfy

$$\frac{\phi^*_{\theta\theta}}{\rho^2} + \frac{\phi^*_\rho}{\rho} + \phi^*_{\rho\rho} + \phi^*_{\zeta\zeta} = 0 \tag{3.6}$$

for $0 \leq \theta \leq 2\pi$ and $(\rho,\zeta)$ in the domain D shown in Fig. 2. The

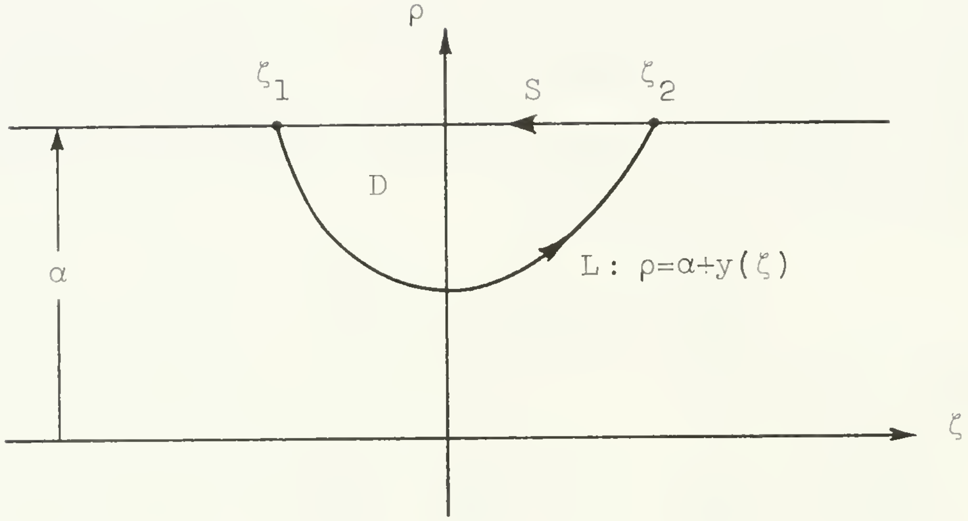

Fig. 2

linearized free surface boundary conditions are

$$-\frac{c\eta^*_\theta(\theta,\zeta)}{\alpha^2\sqrt{gh}} = \phi^*_\rho(\alpha,\theta,\zeta) \tag{3.7}$$

$$F(\theta,\zeta) = -(\alpha^3 gh - c^2)\frac{\eta^*}{\alpha^3 gh} + \frac{c\phi^*_\theta(\alpha,\theta,\zeta)}{\alpha^2\sqrt{gh}} + \text{const.} \tag{3.8}$$

and the wall condition is

$$(3.9) \qquad \phi^*_\rho = \phi^*_\zeta y_\zeta \ .$$

The elimination of $\eta$ from (3.7) and (3.8) gives

$$(3.10) \qquad \alpha(\alpha^3 gh - c^2)\phi^*_\rho(\alpha,\theta,\zeta) + c^2\phi^*_{\theta\theta}(\alpha,\theta,\zeta) = \alpha^2 c\sqrt{gh}\ F_\theta(\theta,\zeta) \ .$$

When the pressure $F(\theta,\zeta)$ and its derivative $F_\theta(\theta,\zeta)$ are small we can expect on physical grounds that the original nonlinear system will possess a unique solution for any finite c, or, what amounts to the same thing, for any mean velocity

$$\overline{c} = \frac{c \iint\limits_D \frac{1}{\rho}\, d\zeta d\rho}{\iint\limits_D d\zeta d\rho}$$

of the undisturbed stream. It also seems reasonable to expect that the solution of the corresponding linear system will provide an approximation to the solution of the nonlinear system. It turns out, however, that the latter expectation cannot be completely fulfilled. We will show that for certain values of $\overline{c}$, which cannot be precluded on physical grounds, a solution of the linear system for a rectangular cross section does not exist; and, in addition, for values of $\overline{c}$ in the neighborhood of such values the linear system can only be satisfied by quantities so large that the linearization is inadmissible.

For the special case of a toroidal channel with a rectangular cross section of breadth $b = \beta h$ and depth h, the disturbance potential $\phi^*$ must satisfy

(3.11) $$\frac{\phi^*_{\theta\theta}}{\rho^2} + \frac{\phi^*_\rho}{\rho} + \phi^*_{\rho\rho} + \phi^*_{\zeta\zeta} = 0$$

for $0 \leq \theta \leq 2\pi$, $0 < \alpha-1 < \rho < \alpha$, $-\frac{\beta}{2} < \zeta < \frac{\beta}{2}$ and the boundary conditions

(3.12) $$\alpha(\alpha^3 gh - c^3)\phi^*_\rho(\alpha,\theta,\zeta) + c^2\phi^*_{\theta\theta}(\alpha,\theta,\zeta) = \alpha^2 c\sqrt{gh}\, F_\theta(\theta,\zeta)$$

(3.13) $$\begin{cases} \phi^*_\rho(\alpha-1,\theta,\zeta) = 0 \\ \phi^*_\zeta(\rho, \ ,\pm \frac{\beta}{2}) = 0 \end{cases}$$

Let us suppose that the pressure is independent of $\zeta$, say given by

(3.14) $$F(\theta,\zeta) = F^*(\theta)$$

so that we have a two-dimensional equation and boundary conditions:

(3.15) $$\rho^2\phi_{\rho\rho}(\rho,\theta) + \rho\phi^*_\rho + \phi^*_{\theta\theta} = 0$$

$$0 \leq \theta \leq 2\pi \ , \quad \alpha-1 < \rho < \alpha$$

(3.16) $$\alpha(\alpha^3 gh - c^2)\phi^*_\rho(\alpha,\theta) + c^2\phi^*_{\theta\theta}(\alpha,\theta) = \alpha^3 c\sqrt{gh}\, F^*(\theta)$$

(3.17) $$\phi^*_\rho(\alpha-1,\theta) = 0 \ .$$

An application of finite Fourier transforms shows that these relations are satisfied by

(3.18) $\phi^*(\rho,\theta)$

$$= \text{const.} + \frac{1}{2\pi}\sum_{j=1}^{\infty} \frac{\alpha^2 c\sqrt{gh}\left[\left(\frac{\rho}{\alpha-1}\right)^j + \left(\frac{\alpha-1}{\rho}\right)^j\right]\int_{-\pi}^{\pi} F^*(x)\sin j\,(x-\theta)dx}{(\alpha^3 gh - c^2)\sinh j\xi - jc^2\cosh j\xi}$$

where

$$c^{\xi} = \frac{\alpha}{\alpha-1}\ , \tag{3.19}$$

and this is the only solution of the problem that exists provided no denominator in the sum is zero. If for some $j = m$, c is such that

$$(\alpha^3 gh - c^2)\sinh m\xi - mc^2\cosh m\xi = 0 \tag{3.20}$$

then a solution exists only if

$$\int_{-\pi}^{\pi} F^*(x)\sin m(x-\theta)dx = 0\ . \tag{3.21}$$

When (3.21) and (3.20) prevail, the solution is not unique and it has the form

(3.22) $\phi^*(\rho,\theta) = \text{const.} + A\left[\left(\frac{\rho}{\alpha-1}\right)^m + \left(\frac{\alpha-1}{\rho}\right)^m\right]\sin m(\theta-B)$

$$+ \frac{1}{2\pi}{\sum_{j=1}^{\infty}}' \frac{\alpha^2 c\sqrt{gh}\left[\left(\frac{\rho}{\alpha-1}\right)^j + \left(\frac{\alpha-1}{\rho}\right)^j\right]\int_{-\pi}^{\pi} F^*(x)\sin j(x-\theta)dx}{(\alpha^2 gh - c^2)\sinh j\xi - jc^2\cosh j\xi}$$

where the prime in the sum indicates that the m-th term is to be deleted, and A and B are arbitrary constants. If $F^*(\theta) \equiv 0$

and (3.20) holds the solution of the homogeneous system is

$$(3.23) \qquad \phi^*(\rho,\theta) = \text{const.} + A[(\frac{\rho}{\alpha-1})^m + (\frac{\alpha-1}{\rho})^m]\sin m(\theta-B) \quad .$$

The values of $c$ which satisfy (3.20) for $m$ an integer $m \geq 1$, that is,

$$(3.24) \qquad c_m = \sqrt{\frac{\alpha^3 gh}{1 + m \coth m\xi}}$$

are values for which $\phi^*$ does not in general exist; and furthermore, the linearization of our problem can be questioned if $c$ is sufficiently close to a $c_m$ because then the magnitude of the $m^{th}$ term in the sum (3.18) is as large as we please if (3.21) does not obtain. The values $c_m$ also define the velocities of the liquid stream for which in the absence of surface pressure there can be steady motions other than the streaming motion, namely those given by

$$\begin{aligned}\phi &= \phi_0 + \phi^* \\ &= -\frac{c\theta}{\sqrt{gh}} + \text{const.} + A[(\frac{\rho}{\alpha-1})^m + (\frac{\alpha-1}{\rho})^m]\sin m\,(\theta-B) \quad .\end{aligned}$$

The mean velocities of the stream corresponding to a $c_m$ are given by

$$(\bar{c}_m)^2 = \left[c_m \int_{\alpha-1}^{\alpha} \frac{d\rho}{\rho}\right]^2 = \frac{\alpha^3 gh\ \ell n^2 \frac{\alpha}{\alpha-1}}{1 + m \coth m\xi}$$

$$= \frac{e^{\frac{3\xi}{2}}}{8 \sinh^3 \frac{\xi}{2} \cdot [\xi + m\xi \coth m\xi]} .$$

It is interesting to notice that as the radius of the channel is allowed to become infinite, that is, as $\alpha \to \infty$, or $\xi \to 0$, we have, for any finite $m$,

$$\lim_{\alpha \to \infty} \bar{c}_m = \sqrt{gh} ,$$

which is the so called critical speed for uniform flow in a straight horizontal channel with rectangular cross section of depth $h$.

For the case of a horizontal channel with rectangular cross section J. J. Stoker [3] has shown that a reason for designating $\sqrt{gh}$ a critical speed can be found in an analysis of certain initial value problems. For example, if a time independent pressure is applied at time $t = 0$ to the surface of a stream running uniformly at any speed $U$ in the channel then, within the framework of the linear theory, equilibrium initial conditions can be used to find the unsteady disturbance potential. It turns out that no matter what the undisturbed equilibrium speed $U$ is, the disturbance is finite for finite $t$. Also, if $U \neq \sqrt{gh}$ the disturbed flow generally tends to a steady flow as $t$ tends to infinity. However, if $U = \sqrt{gh}$ then there can be no steady state; the disturbance potential tends to infinity at all of its space coordinates as $t$ approaches infinity; and this is one reason why $\sqrt{gh}$ is called a critical speed. Stoker [3] notes that even if a dissipative force is introduced so as to cause the unsteady motion to die out the resulting steady motion still becomes unbounded if $U = \sqrt{gh}$. One is thus led to the conclusion that the linear theory fails when the speed $U$ is $\sqrt{gh}$ and for this speed a nonlinear theory is required.

In a similar way it can be shown that if a time independent pressure is applied to the surface of a liquid stream running as defined above in a toroidal channel, then when $c = c_m$ the unsteady disturbance potential tends to infinity for all $\theta$ and $\rho$ as the time tends to infinity.

The values $c_m$ must satisfy

$$c_m^2 = \frac{\alpha^3 gh}{1 + m \coth m\xi} \leq \frac{\alpha^3 gh\ \xi}{\xi + \xi \coth \xi} < \frac{\alpha^3 gh\ \xi}{\xi + 1}$$

and when $\alpha$ is large $\alpha^3 gh\ \xi/(\xi + \xi \coth \xi)$ is close to $\alpha^3 gh\ \xi/(\xi + 1)$. Note that the speed given by

$$c_1^2 = \frac{\alpha^3 gh}{1 + \coth \xi}$$

in the speed of the sinusoidal waves of greatest wave length namely $2\pi\alpha$. It is also interesting to observe that the solutions (3.18) for $\phi^*$ is unique if

$$c^2 > \frac{\alpha^3 gh\,\xi}{\xi + 1} \quad . \tag{3.25}$$

When (3.25) is satisfied the linear theory predicts that in the absence of surface pressure no characteristic sinusoidal surface waves exist.

## 4. Cnoidal Waves

Proofs of the mathematical existence of solitary and cnoidal waves appeared only recently after many scientists had analyzed these waves in numerous papers. Friedrichs and Hyers [4] (1954) and Lavrentiev [5] (1947) proved the mathematical existence of two-dimensional irrotational solitary waves. Littman [6] (1957) proved the existence of two-dimensional irrotational cnoidal waves. These proofs were given about eighty years after Boussinesq [7], [8], (1871, 1872) and Rayleigh [9] (1876) independently of each other presented the first mathematical analysis of the solitary wave. In other words the proofs were given about one hundred ten years after Russell [10], [11] (1838, 1844) wrote the first scientific reports on the solitary wave. Extensive references to the literature can be found in Vol. IX of the Encyclopedia of Physics (Springer-Verlag) which contains the exposition Surface Waves by Wehausen and Laitone.

The existence proofs mentioned above show that if $h$ is the depth of the channel these cnoidal and solitary waves travel with speeds in the neighborhood of $U = \sqrt{gh}$ which as we have already noted is called a critical speed in the linear theory. According to this theory no waves can exist on a running stream subject to zero surface pressure if the speed of the stream is equal to or greater than $\sqrt{gh}$ , whereas for each speed less than this value a sinusoidal wave is possible, and its wavelength tends to infinity as the speed tends to $\sqrt{gh}$ from below. For other characterizations of the critical speed see Stoker's book [3] and the paper by Peters and Stoker [12].

Two-dimensional rotational solitary and cnoidal waves have been investigated by Hunt [13], Peters and Stoker [12], Moiseev [14], Ter-Krikorov [15], Benjamin [16], and Shen [17]. Ter-Krikorov [15] proved the existence of a type of two-dimensional rotational solitary wave by using the method which Friedrichs and Hyers devised for the irrotational case. In a recent paper Peters [1] investigated irrotational and rotational solitary waves in a long straight horizontal channel with arbitrary cross section. Under all circumstances considered in the papers cited above Peters [1] showed that the speeds of the nonlinear waves are predicted by the linear theory to be in the neighborhood of the speeds of the long waves of that theory.

Coupled with what we found in Section 3, the results just mentioned suggest that irrotational cnoidal waves may exist in a toroidal channel with rectangular cross section and travel with speeds in the neighborhood of the critical value

$$\frac{\alpha^3 gh\ \xi}{\xi + \xi \coth \xi}$$

which is nearly the same as

$$\frac{\alpha^3 gh\ \xi}{\xi + 1} \tag{4.1}$$

when $\alpha$ is large. In order to examine this possibility and the more general one of cnoidal waves in a toroidal channel of arbitrary cross section we will use the equations (2.13) - (2.18) which, as we saw in Section 2, are produced by applying a stretching technique to the basic hydrodynamical equations. The equations are

$$u = \frac{-\phi_\sigma}{\rho} \quad , \quad v = \phi_\rho \quad , \quad w = \phi_\zeta \tag{4.2}$$

$$\frac{\varepsilon \phi_{\sigma\sigma}}{\rho^2} + \frac{\phi_\rho}{\rho} + \phi_{\rho\rho} + \phi_{\zeta\zeta} = 0 \tag{4.3}$$

$$\varepsilon\left[\frac{\phi_\sigma^2}{2\rho^2} + \rho + \pi\right] + \frac{\phi_\rho^2}{2} + \frac{\phi_\zeta^2}{2} = C \tag{4.4}$$

(4.5) $$\varepsilon\left[\frac{\phi_\sigma^2}{2\rho^2} + \alpha + \eta\right] + \frac{\phi_\rho^2}{2} + \frac{\phi_\zeta^2}{2} = C$$

(4.6) $$\frac{\varepsilon}{\rho^2}\phi_\sigma\eta_\sigma = \phi_\rho - \phi_\zeta\eta_\zeta$$

(4.7) $$\phi_\rho = \phi_\zeta\eta_\zeta \quad .$$

Equation (4.3) must hold at each point $(\rho,\sigma,\zeta)$ occupied by a liquid particle. Equations (4.5) and (4.6) must be satisfied at the free surface $\rho = \alpha + \eta(\sigma,\zeta)$ while equation (4.7) must be satisfied at the wall of the channel defined by $\rho = \alpha + y(\zeta)$. Equation (4.4) allows us to compute the pressure once $\phi$ is known.

Our object is to examine the consequences of the assumption that $\phi$ and $\eta$ can be expanded in powers of $\varepsilon$, that is,

(4.8) $$\phi(\rho,\sigma,\zeta) = \phi_o + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots$$

(4.9) $$\eta(\sigma,\zeta) = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + \ldots$$

where $\phi_j$ and $\eta_j$ are independent of $\varepsilon$. [We know that expansions of this kind are asymptotically valid for two-dimensional cnoidal waves in a horizontal channel and we can expect them to be valid in the same way for the case of a toroidal channel provided $\alpha$ is large.] The idea is to substitute these expansions in (4.3), (4.5), (4.6), (4.7); and then equate coefficients of like powers of $\varepsilon$. This leads to a sequence of partial differential systems for the determination of $\phi_o, \phi_1, \ldots;$ and $\eta_1, \eta_2, \ldots$ . We need to observe here that the constant C in (4.5) must also be replaced by an expansion in powers of $\varepsilon$. We will write

$$C = C_o + \varepsilon C_1 + \varepsilon^2 C_2 + \varepsilon^3 C_3 + \ldots \, . \tag{4.10}$$

The zeroth order system comes from substituting $\varepsilon = 0$ in (4.3), (4.5), (4.6) and (4.7). We find that $\phi_o(\rho,\sigma,\zeta)$ must satisfy

$$\phi_{o\rho\rho} + \frac{\phi_{o\rho}}{\rho} + \phi_{o\zeta\zeta} = 0 \tag{4.11}$$

in the domain D shown in Fig. 2 in Section 3. This domain is bounded by L: $\rho = \alpha + y(\zeta)$; and a segment of the line $\rho = \alpha$ determined by the points where L intersects $\rho = \alpha$. We assume that L is a piecewise smooth curve which crosses $\rho = \alpha$ in just two points $(\zeta_1, \alpha)$ and $(\zeta_2, \alpha)$ such that $\beta = \zeta_2 - \zeta_1 > 0$ and $\zeta_1 < 0$. The equations (4.5) and (4.6) show that

$$\phi^2_{o\rho}(\alpha,\sigma,\zeta) + \phi^2_{o\zeta}(\alpha,\sigma,\zeta) = C_o \tag{4.12}$$

$$\phi_{o\rho}(\alpha,\sigma,\zeta) = 0 \tag{4.13}$$

and (4.7) shows that the normal derivative $\phi_{on}$ must be zero along L. Since the above requires the normal derivative of

$\phi_o$ to be zero along both L and S it can be proved that

$$\phi_o = F_o(\sigma)$$

is the only solution without singularities at the boundary. Then from equation (4.12) we find

$$C_o = 0 \ .$$

In order to fix $F_o(\sigma)$ we must pass to the next system.

The first order system comes from equating coefficients of $\varepsilon$ in (4.3), (4.5), (4.6) and (4.7). The result of doing this shows that $\phi_1$ must satisfy

$$(4.14) \qquad \phi_{1\rho\rho} + \frac{\phi_{1\rho}}{\rho} + \phi_{1\zeta\zeta} = -\frac{F_o''(\sigma)}{\rho^2}$$

in D, and for S we obtain the conditions

$$(4.15) \qquad \frac{\phi_{o\sigma}^2}{2\alpha^2} + \alpha = C_1$$

$$(4.16) \qquad \phi_{1\rho}(\alpha, \sigma, \zeta) = 0 \ .$$

In addition we find that the normal derivative $\phi_{1n}$ must be zero along L. Equation (4.15) shows that $F_o'(\sigma)$ is a constant for we have

$$\frac{[F_o'(\sigma)]^2}{2\alpha^2} + \alpha = C_1$$

(4.17) $$[F_o'(\sigma)]^2 = (C_1 - \alpha)2\alpha^2 = \frac{\gamma_o^2}{gh} \quad .$$

Consequently $\phi_1$ must satisfy

$$\phi_{1\rho\rho} + \frac{\phi_{1\rho}}{\rho} + \phi_{\zeta\zeta} = 0$$

in D and since $\phi_{1n}$ is zero on L+S we see that

(4.18) $$\phi_1(\sigma,\rho,\zeta) = F_1(\sigma) \quad .$$

The second order system shows that $\phi_2$ must satisfy

(4.19) $$\phi_{2\rho\rho} + \frac{\phi_{2\rho}}{\rho} + \phi_{2\zeta\zeta} = - \frac{F_1''(\sigma)}{\rho^2}$$

in D and $\phi_{2n}$ must vanish on L. Along S the conditions are given by

(4.20) $$\frac{1}{2\alpha^2} (2\phi_{o\sigma}\phi_{1\sigma} - \frac{2\eta_1}{\alpha} \phi_{o\sigma}^2) + \eta_1 = C_2$$

and

(4.21) $$\phi_{2\rho} = \frac{1}{\alpha^2} \phi_{o\sigma}\eta_{1\sigma} \quad .$$

The elimination of $\eta_1$ from (4.20) and (4.21) shows that the normal derivative $\phi_{2\rho}$ along S must be

$$(4.22) \qquad \phi_{2\rho}(\alpha,\sigma,\zeta) = \frac{\gamma_o^2 F_1''(\sigma)}{\alpha(\gamma_o^2 - \alpha^3 gh)} \quad .$$

The constant $\gamma_o$ can be fixed by observing that if $\chi_n$ is the outward normal derivative of $\chi$ at the boundary of D then

$$(4.23) \qquad \iint_D \left[\frac{\partial}{\partial\zeta}(\rho\chi_\zeta) + \frac{\partial}{\partial\rho}(\rho\chi_\rho)\right] d\zeta d\rho = \int_{L+S} \rho\chi_n \, ds$$

where s denotes the arc length along the boundary L+S which is oriented counterclockwise as shown in Fig. 2. This identity is a condition for the existence of a function $\chi$ which has a prescribed normal derivative on the boundary of D and which satisfies

$$\chi_{\rho\rho} + \frac{\chi_\rho}{\rho} + \chi_{\zeta\zeta} = K(\zeta,\rho)$$

or, what is the same equation,

$$\frac{\partial(\rho\chi_\rho)}{\partial\rho} + \frac{\partial}{\partial\zeta}(\rho\chi_\zeta) = \rho K(\zeta,\rho)$$

in D. Such a function does not exist if the assigned normal derivative violates

$$\iint_D \rho K(\zeta,\rho) d\zeta d\rho = \int_{L+S} \rho\chi_n \, ds \; .$$

The application of (4.23) to $\phi_2$ gives

$$-F_1''(\sigma) \iint_D \frac{d\zeta d\rho}{\rho} = \frac{\beta\gamma_o^2 F_1''(\sigma)}{\gamma_o^2 - \alpha^3 gh}$$

or

$$F_1''(\sigma)\left[\iint_D \frac{d\zeta d\rho}{\rho} + \frac{\beta\gamma_o^2}{\gamma_o^2 - \alpha^3 gh}\right] = 0 \tag{4.24}$$

where $\beta h$ is the breadth of the channel at $r = \alpha h$. Equation (4.24) shows that either $F_1''(\sigma) = 0$ or

$$\iint_D \frac{d\zeta d\rho}{\rho} + \frac{\beta\gamma_o^2}{\gamma_o^2 - \alpha^3 gh} = 0 \ . \tag{4.25}$$

This indicates a bifurcation phenomenon. The first alternative leads only to the motion of an irrotational liquid stream and hence it will be ignored here. The second alternative gives

$$\gamma_o^2 = \frac{\alpha^3 gh \iint_D \frac{d\zeta d\rho}{\rho}}{\iint_D \frac{d\zeta d\rho}{\rho} + \beta} \tag{4.26}$$

or

$$[F_o'(\sigma)]^2 = \frac{\alpha^3\mu}{1+\mu} \tag{4.27}$$

where

$$\mu = \frac{1}{\beta} \iint_D \frac{d\zeta d\rho}{\rho} \ . \tag{4.28}$$

It should be noted that if the cross section of the channel is rectangular with depth h then (4.26) gives

$$\gamma_o^2 = \frac{\alpha^3 gh \;\ell n \frac{\alpha}{\alpha-1}}{\ell n \frac{\alpha}{\alpha-1} + 1} \tag{4.29}$$

and if

$$\frac{\alpha}{\alpha-1} = e^{\xi}$$

we have

$$\gamma_o^2 = \frac{\alpha^3 gh\xi}{\xi+1} \tag{4.30}$$

which is the same as (4.1). This fulfills the expectation expressed at the beginning of this section. Equation (4.30) defines the velocity

$$u_o = - \frac{\phi_\sigma}{\rho} = - \frac{1}{\rho} \sqrt{\frac{\alpha^3 gh\xi}{\xi+1}}$$

which is greater in absolute magnitude than any of the velocities cm/ρ associated with the sinusoidal waves of the linear theory discussed in Section 3.

Equation (4.26) defines the mean velocity

$$(4.31) \qquad \bar{u}_o = - \frac{\iint\limits_D \frac{\phi \, d\zeta d\rho}{\rho}}{\iint\limits_D d\zeta d\rho}$$

$$= - \frac{\iint\limits_D \frac{d\zeta d\rho}{\rho}}{\iint\limits_D d\zeta d\rho} \cdot \sqrt{\frac{\alpha^3 gh \iint\limits_D \frac{d\zeta d\rho}{\rho}}{\iint\limits_D \frac{d\zeta d\rho}{\rho} + \beta}} \ .$$

When $\alpha$ is sufficiently large the toroidal channel can be regarded as a horizontal channel of cross section D. For the limit of $\bar{u}_o$ as $\alpha \rightarrow \infty$ we have

$$(4.32) \qquad \lim_{\alpha \rightarrow \infty} \bar{u}_o = - \sqrt{\frac{gh \iint\limits_D d\zeta d\rho}{\beta}}$$

$$= - \sqrt{gd}$$

where d is the mean depth of the horizontal channel. This agrees with a result in a paper by the author [1], namely that $\sqrt{gd}$ is a first order approximation to the speed of cnoidal or solitary waves in a horizontal channel with arbitrary cross section.

It can now be seen from equations (4.19), (4.22) and the result (4.25) that the function $\phi_2$ must satisfy

$$\phi_{2\rho\rho} + \frac{\phi_{2\rho}}{\rho} + \phi_{2\zeta\zeta} = - \frac{F_1''(\sigma)}{\rho^2}$$

for $(\zeta,\rho)$ a point in D;

$$\phi_{2\rho}(\alpha,\sigma,\zeta) = - \frac{1}{\beta\alpha} \iint\limits_{D} \frac{d\zeta d\rho}{\rho} , F_1''(\sigma)$$

and

$$\phi_{2n} = 0$$

along L. This implies that we can write

$$\phi_2 = - \frac{F_1''(\sigma)}{\alpha^2} \psi_2(\rho,\zeta) + F_2(\sigma)$$

where

$$\psi_{2\rho\rho} + \frac{\psi_{2\rho}}{\rho} + \psi_{2\zeta\zeta} = \frac{\alpha^2}{\rho^2} \tag{4.33}$$

for $(\zeta,\rho)$ a point in D;

$$\psi_{2\rho}(a,\zeta) = \alpha\mu \tag{4.34}$$

and

$$\psi_{2n} = 0 \tag{4.35}$$

along L. There is no loss of generality if we make $\psi_2$ unique by specifying its value at some particular point say

$$\psi_2(\alpha,0) = 0 \quad .$$

From equation (4.20) we see that

$$F_1'(\sigma) = \frac{\{[F_o'(\sigma)]^2 - \alpha^3\}]\eta_1}{\alpha F_o'(\sigma)} + \frac{\alpha^2 C_2}{F_o'(\sigma)}$$

or

$$F_1'(\sigma) = -\frac{F_o'(\sigma)\eta_1}{\alpha\mu} + \frac{\alpha^2 C_2}{F_o'(\sigma)} \quad . \tag{4.36}$$

Equation (4.36) shows that $\eta_1 = \eta_1(\sigma)$ is a function of $\sigma$ alone; but to determine $\eta_1(\sigma)$ we must proceed to the next step.

The third order system shows that $\phi_3$ must satisfy

$$\phi_{3\rho\rho} + \frac{\phi_{3\rho}}{\rho} + \phi_{3\zeta\zeta} = -\frac{F_1'''(\sigma)\psi_2(\rho,\zeta)}{\alpha^2\rho^2} - \frac{F_2''(\sigma)}{\rho^2} \tag{4.37}$$

$$= -\frac{F_o'(\sigma)\psi_2(\rho,\zeta)\eta_1'''(\sigma)}{\mu\alpha^3\rho^2} - \frac{F_2''(\sigma)}{\rho^2}$$

in $D$, and $\phi_{3n}$ must vanish on $L$. Along $S$ the requirements are

$$\frac{1}{2\alpha^2}\left[\phi_{1\sigma}^2 + 2\phi_{o\sigma}\phi_{2\sigma} - \frac{4\eta_1}{\alpha}\phi_{o\sigma}\phi_{1\sigma} + \phi_{o\sigma}^2\left(\frac{3\eta_1^2}{\alpha^2} - \frac{2\eta_2}{\alpha}\right)\right] + \eta_2 = C_3 \tag{4.38}$$

and

$$\frac{1}{\alpha^2}\left[\phi_{1\sigma}\eta_{1\sigma} + \phi_{o\sigma}\eta_{2\sigma} - \frac{2\phi_{o\sigma}}{\alpha}\eta_1\eta_{1\sigma}\right] = \phi_{3\rho} + \eta_1\phi_{2\rho\rho} \quad . \tag{4.39}$$

The elimination of $\eta_2$ from (4.38) and (4.39) gives

$$(4.40)\qquad \phi_{3\rho} = -F_o'(\psi_{2\rho\rho} + 2 + 6\mu + 3\mu^2)\,\frac{\eta_1\eta_1'}{\mu\alpha^3} + \frac{2(1+\mu)C_2\eta_1'}{F_o'} - \frac{F_o'\psi_2\eta_1'''}{\alpha^4} - \frac{\mu}{\alpha}F_2'' \quad .$$

Since the normal derivative $\phi_{3n}$ is zero on $L$ , the existence condition (4.23), for $\chi = \phi_3$ , can be written

$$\iint_D \left[\frac{\partial}{\partial\zeta}(\rho\phi_{3\zeta}) + \frac{\partial}{\partial\rho}(\rho\phi_{3\rho})\right]d\zeta d\rho = \alpha\int_{\zeta_1}^{\zeta_2}\phi_{3\rho}(\alpha,\sigma,\zeta)d\zeta \quad .$$

As we can see from (4.37) and (4.40) this condition requires

$$-\frac{F_o'}{\mu\alpha^3}\iint_D \frac{\psi_2(\rho,\zeta)d\zeta d\rho\cdot\eta_1'''}{\rho} - F_2''\iint_D\frac{d\zeta d\rho}{\rho}$$

$$= -F_o'\left[\int_{\zeta_1}^{\zeta_2}\psi_{2\rho\rho}d_\zeta + 2\beta + 6\beta\mu + 3\beta\mu^2\right]\frac{\eta_1\eta_1'}{\mu\alpha^2}$$

$$+\frac{2\beta(1+\mu)\alpha C_2\eta_1'}{F_o'} - \frac{F_o'}{\alpha^3}\int_{\zeta_1}^{\zeta_2}\psi_2 d_\zeta\cdot\eta'''_1 - \mu\beta F_2'' \quad .$$

Since

$$\mu\beta = \iint_D\frac{d\zeta d\rho}{\rho}$$

the terms involving the unknown $F_2$ cancel and we can write

$$(4.41)\qquad \left[\mu\int_{\zeta_1}^{\zeta_2}\psi_2(\alpha,\zeta)d\zeta - \iint_D \frac{\psi_2(\rho,\zeta)d\zeta d\rho}{\rho}\right]\eta_1'''(\sigma)$$

$$= 2\beta\alpha(1+\mu)^2 C_2\eta_1'(\sigma) - \alpha\left[\int_{\zeta_1}^{\zeta_2}\psi_{2\rho\rho}(\alpha,\zeta)d\zeta+2\beta+6\beta\mu+\beta\mu^2\right]\eta_1(\sigma)\eta_1'(\sigma)$$

which is an ordinary nonlinear differential equation for the determination of $\eta_1(\sigma)$ , the first approximation to the surface wave. The coefficient of $\eta_1'''(\sigma)$ can be transformed into

$$m_0 = \mu\int_{\zeta_1}^{\zeta_2}\psi_2(\alpha,\zeta)d\zeta - \iint_D \frac{\psi_2(\rho,\zeta)d\zeta d\rho}{\rho}$$

$$= \frac{1}{\alpha^2}\int_{L+S}\rho\psi_2\psi_{2n}ds - \frac{1}{\alpha^2}\iint_D \psi_2\left[\frac{\partial}{\partial\rho}(\rho\psi_{2\rho}) + \frac{\partial}{\partial\zeta}(\rho\psi_{2\zeta})\right]d\zeta d\rho$$

$$= \frac{1}{\alpha^2}\iint_D \rho[\psi_{2\zeta}^2 + \psi_{2\rho}^2]d\zeta d\rho$$

which shows that $m_0$ is positive. The coefficient of $\eta_1(\sigma)\eta_1'(\sigma)$ is

$$m_1 = -\alpha\left[\int_{\zeta_1}^{\zeta_2}\psi_{2\rho\rho}(\alpha,\zeta)d\zeta+2\beta+6\beta\mu+3\beta\mu^2\right]$$

$$= -\alpha\left\{\int_{\zeta_1}^{\zeta_2}\left[1 - \frac{\psi_{2\rho}(\alpha,\zeta)}{\alpha} - \psi_{2\zeta\zeta}(\alpha,\zeta)\right]d\zeta+2\beta+6\beta\mu+3\beta\mu^2\right\}$$

$$= -\alpha\left\{3\beta+5\beta\mu+3\beta\mu^2 - [\psi_{2\zeta}(\alpha,\zeta_2) - \psi_{2\zeta}(\alpha,\zeta_1)]\right\} \quad .$$

If the curve $L$ intersects $\rho = \alpha$ orthogonally we have

$$\psi_{2\zeta}(\alpha,\zeta_2) = \psi_{2\zeta}(\alpha,\zeta_1) = 0$$

and

$$m_1 = -\alpha(3\beta + 5\beta\mu + 3\beta\mu^2) \quad .$$

If in addition to the above definitions of $m_o$ and $m_1$, we define $m_2$ as

$$m_2 = 2\beta a(1+\mu)^2 C_2$$

the equation (4.41) can be written

$$(4.42) \qquad m_o\eta_1'''(\sigma) = m_1\eta_1(\sigma)\eta_1'(\sigma) + m_2\eta_1'(\sigma) \quad .$$

The difference between this equation and the classical equation for the determination of two-dimensional solitary and cnoidal waves lies only in the values of the coefficients. If we introduce the independent variable $\chi = \alpha\sigma$ and let $\alpha$ tend to infinity equation (4.42) reduces to the equation, given by Peters in [1], which determines solitary and cnoidal waves in a horizontal channel with arbitrary cross section.

If $m_1$ is negative, as it is when $L$ intersects $\rho = \alpha$ orthogonally, equation (4.42) can be written in the form

$$(4.43) \qquad \lambda\eta_1'''(\sigma) = -3\eta_1(\sigma)\eta_1'(\sigma) + \lambda_2\eta_1'(\sigma)$$

where

$$(4.44) \qquad \lambda = \frac{3m_o}{|m_1|}$$

is regarded as known and

$$\lambda_2 = \frac{3m_2}{|m_1|}$$

is regarded as not known since it depends on $C_2$. Equation (4.43) can be solved by using the following well known procedure. An integration of (4.43) gives

$$\lambda\eta_1''(\sigma) = -\frac{3\eta_1^2}{2} + \lambda_2\eta_1 + \frac{\lambda_3}{2}$$

and if we multiply this result by $\eta_1'(\sigma)$ and integrate again we find

$$\lambda[\eta_1'(\sigma)] = -\eta_1^3 + \lambda_2\eta_1^2 + \lambda_3\eta_1 + \lambda_4$$

or

$$(4.45) \qquad \lambda[\eta_1'(\sigma)]^2 = (q_1-\eta_1)(q_2+\eta_1)(q_3+\eta_1) \quad .$$

Since $\lambda$ is positive and $\eta_1'(\sigma)$ must be real when $\eta_1 = 0$ the product $q_1q_2q_3$ must be positive. At least one of the $q_i$'s is positive and we can take $q_1 > 0$. The slope $\eta_1'(\sigma)$ must vanish at some value of $\eta_1$ other than $\eta_1 = q_1$ for if it did not $\eta_1$ would have to become infinite which is physically impossible. If $\rho = \alpha$ gives the mean value of the free surface then we cannot have $q_2 < 0$ and $q_3 < 0$ for if this were the case $\eta_1$ would have to lie between the extreme values of $q_1$, $-q_2$, $-q_3$ and thus be positive which is again not physically possible. It follows then that $q_2 > 0$ with $q_3 > 0$ and we can take $q_3 \geq q_2$.

If we introduce the new variable $\Omega$ through

$$(4.46) \qquad \eta_1 = q_1 \cos^2 \Omega - q_2 \sin^2 \Omega$$

equation (4.45) becomes

$$(4.47) \qquad d\sigma = \sqrt{\frac{4\lambda}{q_1+q_3}} \cdot \frac{d\Omega}{\sqrt{1-k^2 \sin^2 \Omega}}$$

where

$$k = \sqrt{\frac{q_1+q_2}{q_1+q_3}} \quad .$$

If we take $\sigma = 0$ when $\eta_1 = q_1$ then $\Omega = 0$ when $\sigma = 0$ and

$$(4.48) \qquad \sigma = \sqrt{\frac{4\lambda}{q_1+q_3}} \int_0^{\Omega} \frac{dt}{\sqrt{1-k^2 \sin^2 t}} \quad .$$

This defines the amplitude function

$$\Omega = \mathrm{am}\left( \sigma \sqrt{\frac{q_1+q_3}{4\lambda}} \, , \, k \right) \quad .$$

This function and the functions

$$\cos \Omega = \cos \operatorname{am} \left( \sigma \sqrt{\frac{q_1+q_3}{4\lambda}} , k \right)$$

$$= \operatorname{cn} \left( \sigma \sqrt{\frac{q_1+q_3}{4\lambda}} , k \right)$$

$$\sin \Omega = \sin \operatorname{am} \left( \sigma \sqrt{\frac{q_1+q_3}{4\lambda}} , k \right)$$

$$= \operatorname{sn} \left( \sigma \sqrt{\frac{q_1+q_3}{4\lambda}} , k \right)$$

$$1 - k^2 \sin^2 \Omega = \operatorname{dn} \left( \sigma \sqrt{\frac{q_1+q_3}{4\lambda}} , k \right)$$

are called Jacobi elliptic functions and in terms of the cn function we have

$$\eta_1(\sigma) = -q_2 + (q_1+q_2)\operatorname{cn}^2 \left( \sigma \sqrt{\frac{q_1+q_3}{4\lambda}} , \sqrt{\frac{q_1+q_2}{q_1+q_3}} \right) . \tag{4.49}$$

This represents a periodic profile whose period $\sqrt{\varepsilon}\, T$ is

$$\sqrt{\varepsilon}\, T = \sqrt{\frac{4\lambda}{q_1+q_3}} \int_0^{\pi} \frac{dt}{\sqrt{1-k^2 \sin^2 t}}$$

$$= 2 \sqrt{\frac{4\lambda}{q_1+q_3}} \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2 \sin^2 t}} .$$

Boussinesq [18] was the first to indicate the existence of two-dimensional waves of the type (4.49) and the name cnoidal waves, obviously suggested by the notation cn for the Jacobi elliptic function and the term "sinusoidal", was introduced by Korteweg and De Vries [19].

In addition to

$$k = \sqrt{\frac{q_1+q_2}{q_1+q_3}}$$

it is convenient to introduce the quantities

$$q = \varepsilon h q_1$$

and

$$\ell = 1 + \frac{q_3}{q_1} \quad .$$

In terms of these quantities we have

$$\lambda_2 = \frac{3m_2}{|m_1|} = \frac{6\alpha\beta(1+\mu)^2 C_2}{|m_1|} = q_1 - (q_2+q_3)$$

so that

$$C_2 = \frac{q|m_1|}{\varepsilon h} \frac{[3-\ell(k^2+1)]}{6\alpha\beta(1+\mu)^2} \quad . \tag{4.50}$$

Equation (4.49) takes the form

$$\eta_1(\sqrt{\varepsilon}\,\theta) = \frac{q}{\varepsilon h}\left[(1-\ell k^2) + \ell^2 k^2 \operatorname{cn}^2\left(\frac{\theta}{2}\sqrt{\frac{q\ell}{h\lambda}}\ ,\ k\right)\right] \quad . \tag{4.51}$$

As a function of $\theta$ the period of $\eta_1(\sqrt{\varepsilon}\,\theta)$ is given by

$$T = 2\sqrt{\frac{h\lambda}{q\ell}} \int_0^{\pi} \frac{dt}{\sqrt{1-k^2 \sin^2 t}} \quad . \tag{4.52}$$

Since we require the mean depth of $\eta_1(\sqrt{\varepsilon}\,\theta)$ to be zero we must have

$$\int_0^T \left[(1-\ell k^2) + \ell^2 k^2 \mathrm{cn}^2\left(\frac{\theta}{2}\sqrt{\frac{q\ell}{h\lambda}}\,,\ k\right)\right] d\theta = 0$$

and this leads to the relation

$$(4.53) \qquad (1-\ell)\int_0^\pi \frac{d\Omega}{\sqrt{1-k^2\sin^2\Omega}} + \ell\int_0^\pi \sqrt{1-k^2\sin^2\Omega}\; d\Omega = 0 \quad .$$

In terms of the tabulated complete elliptic integrals of the first and second kinds, namely

$$F_1(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}} \qquad E_1(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}\; dt$$

the equations (4.52) and (4.53) are respectively

$$(4.54) \qquad \frac{T}{4}\sqrt{\frac{q\ell}{h\lambda}} = F_1(k)$$

and

$$(4.55) \qquad (1-\ell)F_1(k) + \ell\, E_1(k) = 0 \quad .$$

The quantities $k$ and $\ell$ can be determined from these equations when $q$, the height of the crest of the first order wave above the mean free surface, and the period $T$ are given.

The depth of the trough below the mean free surface is

$$\check{q} = \varepsilon h q_2 = q(\ell k^2 - 1) \quad .$$

This shows that beside the combination $\{q, T\}$ the combinations $\{q, \check{q}\}$ and $\{\dot{q}, T\}$ also suffice to determine $\eta_1$.

The pressure equation (4.4) can be written as

$$\pi(\rho,\sigma,\zeta) = -\rho - \frac{\phi_\sigma^2}{2\rho^2} + \frac{C}{\varepsilon} - \frac{1}{2\varepsilon}\left(\phi_\rho^2 + \phi_\zeta^2\right)$$

and by using the foregoing results this is the same as

$$\pi(\rho,\sigma,\zeta) = -\rho - \frac{\phi_{o\sigma}^2}{2\rho^2} + C_1 + \varepsilon\left[C_2 - \frac{\phi_{o\sigma}\,\phi_{1\sigma}}{\rho^2}\right] + o(\varepsilon)$$

where the notation $G(\varepsilon) = o(\varepsilon)$ means $\lim\limits_{\varepsilon \to o} \frac{G(\varepsilon)}{\varepsilon} = 0$.

## 5. Summary and Discussion

In terms of the original variables our theory up to this point shows the following. The equation of the free surface is

$$(5.1) \qquad r = a + f = a + h\eta = a + \varepsilon h \eta_1 + o(\varepsilon)$$

$$= a + q\left[(1-\ell k^2) + \ell k^2 \mathrm{cn}^2\left(\frac{\theta}{2}\sqrt{\frac{q\ell}{h\lambda}}\ ,\ k\right)\right] + o(q)$$

where cn denotes the Jacobi elliptic function. The definitions of the parameters which appear in (5.1) and succeeding equations are listed below.

The clockwise tangential speed is given by

$$v_1 = \sqrt{gh}\, u = -\sqrt{gh}\, \frac{\phi_\sigma}{\rho} = -\frac{\sqrt{gh}}{r} \cdot h\left(\phi_{0\sigma}+\epsilon\phi_{1\sigma}\right) + o(\epsilon)$$

$$= -\frac{\sqrt{gh}}{r} \cdot h[F_0'(\sigma) + \epsilon F_1'(\sigma)] + o(\epsilon)$$

$$= -\frac{\sqrt{gh}}{r} \cdot h\left[F_0'(\sigma) + \frac{\epsilon C_2\alpha^2}{F_0'(\sigma)} - \frac{\epsilon F_0'(\sigma)\eta_1(\sigma)}{\alpha\mu}\right] + o(\epsilon)$$

and from (4.27), (4.50) and (4.51) this is

$$(5.2)\quad v_1 = -\frac{a}{r}\sqrt{gh} \cdot \sqrt{\frac{\alpha\mu}{1+\mu}} \left\{ 1 + \frac{q|m_1|[3-\ell(k^2+1)]}{h\ 6\alpha^2\mu\beta(1+\mu)} - \frac{q}{h\alpha\mu}\left[(1-\ell k^2)+\ell k^2 \mathrm{cn}^2\left(\frac{\theta}{2}\ \frac{q\ell}{h\lambda}, k\right)\right] \right\} + o(q).$$

The radial speed is given by

$$v_2 = \frac{\sqrt{gh}}{\sqrt{\epsilon}}\, v = \frac{\sqrt{gh}}{\sqrt{\epsilon}} \cdot \phi_\rho = \frac{\sqrt{gh}}{\sqrt{\epsilon}}\,[\epsilon^2\phi_{2\rho}+o(\epsilon^2)]$$

$$= -\sqrt{gh}\,\epsilon^{3/2}\,\frac{F_1''(\sigma)}{\alpha^2}\,\psi_{2\rho}(\rho,\zeta) + o(\epsilon^2)]$$

$$= \frac{\sqrt{gh}}{\alpha^3\mu}\,\epsilon^{3/2}\,F_0'(\sigma)\eta_1'(\sigma)\psi_{2\rho}(\rho,\zeta) + o(\epsilon^{3/2}) \quad ,$$

that is,

$$(5.3)\quad v_2 = \frac{\sqrt{gh}}{\alpha^2\mu}\sqrt{\frac{\alpha\mu}{1+\mu}} \cdot \frac{k^2}{h} \cdot \frac{1}{\sqrt{h\lambda}} \cdot (q\ell)^{3/2} \cdot \psi_{2\rho}(\rho,\zeta) \cdot$$

$$\cdot \operatorname{cn}\left(\frac{\theta}{2}\sqrt{\frac{q\ell}{h\lambda}}, k\right) \cdot \operatorname{sn}\left(\frac{\theta}{2}\sqrt{\frac{q\ell}{h\lambda}}, k\right) \cdot$$

$$\cdot \operatorname{dn}\left(\frac{\theta}{2}\sqrt{\frac{q\ell}{h\lambda}}, k\right) + o(q^{3/2}) \quad .$$

The axial speed is given by

$$v_3 = \frac{\sqrt{gh}}{\sqrt{\varepsilon}} \cdot w = \frac{\sqrt{gh}}{\sqrt{\varepsilon}}\phi_\zeta = \frac{\sqrt{gh}}{\sqrt{\varepsilon}}[\varepsilon^2\phi_{2\zeta} + o(\varepsilon^2)]$$

which is

$$(5.4)\quad v_3 = \frac{\sqrt{gh}}{\alpha^2\mu} \cdot \sqrt{\frac{\alpha\mu}{1+\mu}} \cdot \frac{k^2}{h} \cdot \frac{1}{\sqrt{h\lambda}}(q\ell)^{3/2}\,\psi_{2\zeta}(\rho,\zeta) \cdot$$

$$\cdot \operatorname{cn}\frac{\theta}{2}\left(\sqrt{\frac{q\ell}{h\lambda}}, k\right) \operatorname{sn}\frac{\theta}{2}\left(\sqrt{\frac{q\ell}{h\lambda}}, k\right) \cdot \operatorname{dn}\left(\frac{\theta}{2}\sqrt{\frac{q\ell}{h\lambda}}, k\right) + o(q^{3/2}) .$$

For the pressure we have

$$p = \delta gh\pi = \delta gh \left\{ \begin{array}{l} -\rho - \dfrac{(F_o')^2}{2\rho^2} + c_1 \\ + \varepsilon[c_2 - \dfrac{F_o'F_1'}{\rho^2}] \end{array} \right\} + o(\varepsilon)$$

or

$$(5.5)\quad p = \delta g \left\{ \begin{matrix} a - r + \left(1 - \frac{a^2}{r^2}\right) \frac{q|m_1|[3-\ell(k^2+1)]}{6\alpha\beta(1+\mu)^2} \\ + \frac{a\mu}{2(1+\mu)}\left(1 - \frac{a^2}{r^2}\right) \\ + \frac{a^2}{r^2} q[(1-\ell k^2)+\ell k^2 \mathrm{cn}^2(\frac{\theta}{2}\sqrt{\frac{q\ell}{h\lambda}}, k)] \end{matrix} \right\} + o(q).$$

For convenience we restate the definitions of the terms in the above formulas which do not have obvious meanings:

q denotes the height above the mean free surface, r = a, of a crest of the first order wave.

k denotes a number which is related to the period T of the wave. This period is given by

$$T = 2\sqrt{\frac{h\lambda}{q\ell}} \int_0^{\pi} \frac{dt}{\sqrt{1-k^2\sin^2 t}} .$$

The equation

$$(1-\ell)\int_0^{\pi} \frac{d\Omega}{\sqrt{1-k^2\sin^2\Omega}} + \ell \int_0^{\pi} \sqrt{1-k^2\sin^2\Omega}\; d\Omega = 0$$

relates the quantity $\ell$ to k. It insures that the mean free surface is r = a. This equation with the equation for the period fixes $\ell$ and k when q and T are prescribed.

$\lambda$ is defined by the ratio

$$\lambda = \frac{3\ m_0}{|m_1|}$$

where

$$m_0 = \frac{1}{\alpha^2} \iint_D \rho(\psi_{2\zeta}^2 + \psi_{2\rho}^2)\ d\zeta\ d\rho$$

and

$$m_1 = -\alpha\,\beta[3+5\mu + 3\mu^2 - \frac{1}{\beta}\int_{\zeta_1}^{\zeta_2} \psi_{2\zeta\zeta}(\alpha,\xi)\ d\zeta].$$

D stands for a dimensionless radial cross section of the equilibrium position of the liquid. It is bounded by the segment S defined by $\rho = \alpha,\ \zeta_1 < \zeta < \zeta_2$; and the channel wall curve L: -- $\rho = \alpha + \frac{1}{h}\,Y(\frac{x_3}{h}) = \alpha + y(\zeta)$.

$\psi_2(\rho,\zeta)$ denotes the solution of the system

$$\psi_{2\rho\rho} + \frac{\psi_{2\rho}}{\rho} + \psi_{2\zeta\zeta} = \frac{\alpha^2}{\rho^2} \qquad (\rho,\zeta) \text{ in D}$$

$$\psi_{2\rho}(\alpha,\zeta) = \alpha\mu \qquad (\alpha,\zeta) \text{ on S}$$

$$\psi_{2n}(\rho.\zeta) = 0 \qquad (\rho,\zeta) \text{ on L}$$

$$\psi_2(\alpha.0) = 0\ .$$

$\mu$ is defined by the ratio

$$\mu = \frac{1}{\beta} \iint_D \frac{d\zeta\ d\rho}{\rho}$$

where $\beta = \zeta_2 - \zeta_1$.

We do not give the details here but the following statement can be proved. In the limit as the radius $\alpha h$ of the free surface is allowed to tend to infinity and simultaneously cause the wavelength $h\alpha T$ to become infinite, the formulas listed above reduce to those found by the author in [1] for irrotational solitary waves in a horizontal channel with arbitrary cross section. We note that if $h\alpha T$ is infinite then $k = 1$ which implies $q_2 = q_3$. For $k = 1$ we have

$$\cos \Omega = \operatorname{cn}\left(\sigma \sqrt{\frac{q_1+q_3}{4\lambda}}\,, 1\right) = \operatorname{sech} \sigma \sqrt{\frac{q_1+q_3}{4\lambda}}$$

and consequently

$$\eta_1(\sigma) = - q_2 + (q_1 + q_2) \operatorname{sech}^2 \frac{\sigma}{2} \sqrt{\frac{q_1+q_3}{\lambda}} \quad .$$

If we impose the condition $\eta_1(\infty) = 0$, as we may without loss of generality, then $q_2 = 0$ from which it follows that $q_3 = 0$, $\ell = 1$, and

$$\eta_1(\sigma) = q_1 \operatorname{sech}^2 \frac{\sigma}{2} \sqrt{\frac{q_1}{\lambda}}$$

which gives the profile of the solitary wave.

When the cross section of the toroidal channel is a rectangle of breadth $b = \beta h$ and depth $h$ the quantity $\mu$ is

$$\mu = \ln \left(\frac{\alpha}{\alpha - 1}\right)$$

and $\psi_2$ must satisfy

$$\psi_{2\rho\rho} + \frac{\psi_{2\rho}}{\rho} + \psi_{2\zeta\zeta} = \frac{\alpha^2}{\rho^2}, \qquad \begin{array}{l} \alpha-1 < \rho < \alpha, \\ -\frac{\beta}{2} < \zeta < \frac{\beta}{2}, \end{array}$$

$$\psi_{2\rho}(\alpha,\zeta) = \ln(\frac{\alpha}{\alpha-1})$$

$$\psi_{2\rho}(\alpha-1,\zeta) = 0$$

$$\psi_{2\zeta}(\rho, \pm\frac{\beta}{2}) = 0$$

$$\psi_2(\alpha,0) = 0 .$$

For this case it is easy to see that

$$(5.6) \quad \psi_2(\rho,\zeta) = \psi_2(\rho) = \frac{\alpha^2}{2}\,[\ln^2(\frac{\rho}{\alpha-1}) - \ln^2(\frac{\alpha}{\alpha-1})] ,$$

and that the motion of the liquid in the channel is two-dimensional. Associated with (5.6) we have

$$m_0 = \frac{\beta}{\alpha^2}\int_{\alpha-1}^{\alpha} \rho\, \psi_{2\rho}^2\, d\rho = \frac{\beta\alpha^2}{3}\ln^3(\frac{\alpha}{\alpha-1})$$

$$m_1 = -\,\alpha\beta\,[3 + 5\ln(\frac{\alpha}{\alpha-1}) + 3\ln^2(\frac{\alpha}{\alpha-1})]$$

so that $\lambda$ is given by

$$\lambda = \frac{\alpha\ln^3(\frac{\alpha}{\alpha-1})}{3 + 5\ln(\frac{\alpha}{\alpha-1}) + 3\ln^2(\frac{\alpha}{\alpha-1})} .$$

For the case of a rectangular cross section the mean tangential speed over a period is

$$\tilde{v}_1 = \frac{\int_o^T \int_{a-h}^{a+f} v_1 \, r \, dr \, d\theta}{\int_o^T \int_{a-h}^{a+f} r \, dr \, d\theta} .$$

$$= \frac{-\sqrt{gh}\sqrt{\frac{\alpha\mu}{1+\mu}}}{1 - \frac{h}{2a}} \left\{ 1 + \frac{q}{h} \frac{[3+5\mu+3\mu^2][3-\ell(k^2+1)]}{6\,\alpha\mu(1+\mu)} \right\} + o(q)$$

where

$$\mu = \ln \frac{\alpha}{\alpha-1} = \ln \frac{a}{a-h} .$$

In accordance with one of Stokes' definitions for the speed of a two-dimensional undulatory wave, $|\tilde{v}_1|$ can be regarded as the velocity of a cnoidal wave on a cylinder. In the limit as $a \to \infty$, $|\tilde{v}_1|$ reduces to

$$\lim_{a \to \infty} |\tilde{v}_1| = \sqrt{gh}\,(1 + \frac{q}{2h})$$

which is the well known formula for the approximate speed of a two-dimensional solitary wave.

From the two-dimensional theory of Section 3 the wave length of the longest characteristic sinusoidal wave is $2\pi a$ and the corresponding free surface wave profile is given by

$$r = a + A_1 \cos\theta .$$

This represents a cylindrical surface whose cross section has a single crest and a single trough, but the height of the crest above the mean free surface is the same as the depth of the trough. For a toroidal channel with rectangular cross section the cnoidal wave of length $2\pi a$ defined by (5.1) is also a cylindrical wave with a single crest and a single trough but for this wave the height of the crest is not necessarily the same as the depth of the trough. Generally speaking, the part of the cnoidal wave above the mean free surface is narrower in cross section than the part below. Furthermore, if a is sufficiently large the depth of the trough of the cnoidal wave can be made as small as we please while the height of the crest is kept finite. These observations can be used to justify the interpretation of the cnoidal wave of length $2\pi a$ as a two-dimensional "solitary" wave on a cylindrical liquid body.

If the cross section of the toroidal channel is not rectangular, $\psi_2(\rho,\zeta)$ depends on both $\rho$ and $\zeta$ and since each of r, $v_1$, $v_2$, $v_3$ and p depend on $\psi_2$ the motion has three-dimensional aspects. To the order of approximation given above, the height of the free surface does not depend on $\zeta$, that is, the height is constant in a lateral direction across the channel. It should be remarked, however, that this is no longer true for higher order approximations. The next term in the development of the wave surface depends on $\eta_2$ and (4.38) shows that

$$\eta_2 = -\frac{(1+\mu)(1+3\mu)\eta_1^2(\sigma)}{2\alpha\,\mu} + \frac{c_2(1+\mu)(1+2\mu)\eta_1(\sigma)}{\alpha\,\mu}$$

$$-\frac{(1+\mu)^2 c_2^2}{2\,\alpha\,\mu} + (1+\mu)c_3 - \frac{\sqrt{\alpha\mu(1+\mu)}}{\alpha}\,F_2'(\sigma)$$

$$-\frac{1}{\alpha^2}\,\eta_1''(\sigma)\,\psi_2(\rho,\zeta)\ .$$

The last term in this expression modifies the amplitude of $\eta_2$ in the $\zeta$ direction. Therefore the approximation

$$f = h\varepsilon\,\eta_1 + h\,\varepsilon^2\eta_2 + o(\varepsilon^2)$$

furnishes an example of a three-dimensional cnoidal wave.

In conclusion we wish to point out that although the foregoing analysis is confined to irrotational motion it can be extended to study motions which may exist as bifurcations from the motion of a rotational liquid stream.

# References

[1] Peters, A. S., Irrotational and Rotational Solitary Waves in a Channel of Arbitrary Cross Section, N.Y.U., Courant I. M. S., Report No. 345, Jan. 1966

[2] Friedrichs, K. O., On the Derivation of the Shallow Water Theory. Appendix to Stoker, J. J., The Formation of Breakers and Bores, Comm. Pure and Appl. Math., Vol. I, No. 1, 1948, pp. 81-87.

[3] Stoker, J. J., Water Waves, Interscience Publishers, New York, 1957, pp. 207-218.

[4] Friedrichs, K. O. and Hyers, D. H., The Existence of Solitary Waves, Comm. Pure and Appl. Math., Vol. VII, No. 3, 1954, pp. 517-550.

[5] Lavrentiev, M., On the Theory of Long Waves, Akad. Nauk. Ikrain. R.S.R., Zbornik Prac. Inst. Mat., No. 5, 1946, No. 8, 1947, pp. 13-69. (In Ukranian.) A.M.S. Translation No. 102.

[6] Tittman, W., On the Existence of Periodic Waves Near Critical Speed, Comm. Pure and Appl. Math., Vol. X, No. 2, 1957, pp. 241-269.

[7] Boussinesq, J., Theorie de l'Intumescence Liquide Appelée Onde Solitaire ou de Translation se Propageant dans un Canal Rectangulaire, C. R. Acad. Sci., Paris 72, 1871, pp. 755-759.

[8] Boussinesq, J., Theorie des Ondes et des Remous qui se Propageant le Long d'un Canal Rectangulaire Horizontal, en Communiquant au Liquide Contenu dans ce Canal des Vitesses Sensiblement Paveilles de la Surface au Fond, J. Math. Pures Appl. (2) 17, 1872, pp. 55-108.

[9] Rayleigh, J. W. S., Tord, On Waves, Phil. Mag. (5), Vol. 1, 1876, pp. 257-279.

[10] Russell, J. S., Report of the Committee on Waves, Report of the seventh meeting of the British Association for the Advancement of Science, Liverpool, 1837, pp. 417-496. London: John Murray, 1838.

[11] Russell, J. S., Report on Waves, Report of the fourteenth meeting of the British Association for the Advancement of Science, York, 1844, pp. 311-390. London: John Murray 1846.

[12] Peters, A. S., and Stoker, J. J., *Solitary Waves in Liquids Having Non-Constant Density*, Comm. Pure and Appl. Math., Vol. XIII, No. 1, 1960, pp. 115-164.

[13] Hunt, J. N., *Gravity Waves in Flowing Water*, Proc. Roy. Soc. London, Ser. A, Vol. 231, 1955, pp. 496-504.

[14] Moiseev, N. N., *On the Theory of Waves in a Fluid with Vorticity*, Zh. Prikl. Mekh. i Tekhn. Fiz., No. 3, 1960, pp. 81-89.

[15] Ter-Krikorov, A. M., *The Solitary Wave on the Surface of a Fluid with Vorticity*, Zh. Vychisl. Mat. i Mat. Fiz., Vol. I, 1961, pp. 1077-1088.

[16] Benjamin, T. B., *The Solitary Wave on a Stream with an Arbitrary Distribution of Vorticity*, Journal of Fluid Mech., Vol. 12, Part 1, 1962, pp. 97-116.

[17] Shen, M. C., *Solitary Waves in Running Gases*, N.Y.U., Courant I.M.S., Report No. 341, May 1965.

[18] Boussinesq, J., *Essai sur la Theories des Eaux Courantes*, Mem. Divers Savants Acad. Sci., Vol. 32, 1891, p. 392.

[19] Korteweg, D. J. and De Vries, G., *On the Change of Form of Long Waves Advancing in a Rectangular Canal and on a New Type of Long Stationary Waves*, Phil. Mag., Ser. 5, Vol. 39, 1895, pp. 422-443.

Security Classification

**DOCUMENT CONTROL DATA - R&D**

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| | |
|---|---|
| 1 ORIGINATING ACTIVITY *(Corporate author)*<br>Courant Institute of Mathematical Sciences<br>New York University | 2a REPORT SECURITY CLASSIFICATION<br>not classified |
| | 2b GROUP<br>none |

3. REPORT TITLE

Cnoidal Waves in a Toroidal Channel

4 DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report February 1966

5 AUTHOR(S) *(Last name, first name, initial)*

Peters, Arthur S.

| | | |
|---|---|---|
| 6 REPORT DATE<br>February 1966 | 7a. TOTAL NO OF PAGES<br>51 | 7b. NO. OF REFS<br>19 |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO.<br>Nonr-285(55) | 9a. ORIGINATOR'S REPORT NUMBER(S)<br>IMM 347 |
| b. PROJECT NO.<br>NR 062-160 | |
| c. | 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*<br>none |
| d. | |

10 AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

| | |
|---|---|
| 11 SUPPLEMENTARY NOTES<br>none | 12 SPONSORING MILITARY ACTIVITY<br>U.S. Navy, Office of Naval Research<br>207 West 24th St., New York. N.Y. |

13 ABSTRACT

This report is concerned with cnoidal waves in a liquid or gaseous ocean confined to a channel along a parallel of latitude on a sphere.

DD FORM 1 JAN 64 1473

Security Classification

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| | | | | | | |

## INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
______________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
______________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
______________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, *or (U)*

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

Nonr-285(55)

# APPROVED DISTRIBUTION LIST

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 438 (3)

Commanding Officer
Office of Naval Research
Branch Office
219 S. Dearborn Street
Chicago, Illinois 60604 (1)

Commanding Officer
Office of Naval Research
Branch Office
207 West 24th St.
New York 11, N.Y. (1)

Commanding Officer
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, Calif. (1)

Commanding Officer
Office of Naval Research
Branch Office
1076 Mission Street
San Francisco, California (1)

Commanding Officer
Office of Naval Research
Navy No. 100, Fleet Post
Office
New York, New York (5)

Director
Naval Research Laboratory
Washington 25, D. C.
Attn: Code 2021 (6)

Defense Documentation Center
Cameron Station
Alexandria, Va. 22314 (20)

Professor W.R. Sears
Director
Graduate School of Aeronautical
Engineering
Cornell University
Ithaca, New York (1)

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Attn: Research and Development Division (1)

Office of Ordnance Research
Department of the Army
Washington 25, D. C. (1)

Headquarters
Air Research and Development
Command
United States Air Force
Andrews Air Force Base
Washington 25, D. C. (1)

Director of Research
National Advisory Committee
for Aeronautics
1724 F Street, Northwest
Washington 25, D. C. (1)

Director
Langley Aeronautical Laboratory
National Advisory Committee
for Aeronautics
Langley Field, Virginia (1)

Director
National Bureau of Standards
Washington 25, D. C.
Attn: Fluid Mechanics Section (1)

Professor R. Courant
Courant Institute of
Mathematical Sciences, N.Y.U.
251 Mercer St.
New York 12, N.Y. (1)

Professor G. Kuerti
Department of Mechanical
Engineering
Case Institute of Technology
Cleveland, Ohio (1)

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)
Code 420 Preliminary Design (1)

Commander
Naval Ordnance Test Station
3202 E. Foothill Blvd.
Pasadena, Calif. (1)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C.
Attn: Hydromechanics Lab. (1)
Hydrodynamics Div. (1)
Library (1)
Ship Division (1)

California Institute of Technology
Hydrodynamic Laboratory
Pasadena 4, California (1)

Professor A.T. Ippen
Hydrodynamics Laboratory
Massachusetts Institute of Technology
Cambridge 39, Mass. (1)

Dr. Hunter Rouse, Director
Iowa Institute of Hydraulic Research
State University of Iowa
Iowa City, Iowa (1)

Stevens Institute of Technology
Experimental Towing Tank
711 Hudson Street
Hoboken, New Jersey (1)

Dr. G. H. Hickox
Engineering Experiment Station
University of Tennessee
Knoxville, Tennessee (1)

Dr. L. G. Straub
St. Anthony Falls Hydraulic Laboratory
University of Minnesota
Minneapolis 14, Minn. (1)

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 416 (1)
Code 460 (1)

Chief, Bureau of Yards and Docks
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)

Hydrographer
Department of the Navy
Washington 25, D. C. (1)

Director
Waterways Experiment Station
Box 631
Vicksburg, Mississippi (1)

Office of the Chief of Engineers
Department of the Army
Gravelly Point
Washington 25, D. C. (1)

Beach Erosion Board
U.S. Army Corps of Engineers
Washington 25, D. C. (1)

Commissioner
Bureau of Reclamation
Washington 25, D. C. (1)

Dr. G. H. Keulegan
National Hydraulic Laboratory
National Bureau of Standards
Washington 25, D. C. (1)

Brown University
Graduate Division of Applied Mathematics
Providence 12, Rhode Island (1)

California Institute of Technology
Hydrodynamics Laboratory
Pasadena 4, California
Attn: Professor M. S. Plesset (1)
Professor V.A. Vanoni (1)

DISTRIBUTION LIST (CONT.)

Professor M. L. Albertson
Department of Civil Engineering
Colorado A. + M. College
Fort Collins, Colorado (1)

Professor G. Birkhoff
Department of Mathematics
Harvard University
Cambridge 38, Mass. (1)

Massachusetts Institute of
Technology
Department of Naval Architecture
Cambridge 39, Mass. (1)

Dr. R. R. Revelle
Scripps Institute of Oceanography
La Jolla, California (1)

Stanford University
Applied Mathematics and
Statistics Laboratory
Stanford, California (1)

Professor H.A. Einstein
Department of Engineering
University of California
Berkeley 4, Calif. (1)

Dean K. L. Schoenherr
College of Engineering
University of Notre Dame
Notre Dame, Indiana (1)

Director
Woods Hole Oceanographic
Institute
Woods Hole, Mass. (1)

Hydraulics Laboratory
Michigan State College
East Lansing, Michigan
Attn: Professor H.R. Henry (1)

Professor J.W. Johnson
Fluid Mechanics Laboratory
University of California
Berkeley 4, Calif. (1)

APR 26 1966

DATE DUE

GAYLORD PRINTED IN U.S.A.

NYU c.2
IMM-
347 Peters

Cnoidal waves in a toroidal
channel.

NYU c.2

NYU c.2
IMM-
347 Peters
AUTHOR
Cnoidal waves in a toroidal
TITLE
channel.

| DATE DUE | BORROWER'S NAME |
|---|---|
| | |
| | |
| | |

N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York 12, N. Y.